Crossing numbers and biplanar crossing numbers

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A *biplanar drawing of a graph* G means partitioning the edge set of the graph into two graphs, G_1 and G_2, and drawing G_1 and G_2 in two disjoint planes. A graph is *biplanar*, if it admits a biplanar drawing without edge crossings, i.e. the graph has thickness at most 2 (Beineke 1997). Biplanar drawings have obvious significance for VLSI, since such a drawing has physical realization using two sides of a chip, when vertices are present on both sides of the chip (Owens 1970). Unlike planarity, testing graph biplanarity is NP-complete (Mansfield 1983).

In this talk we study the biplanar crossing number $cr_2(G)$ of the graph G. Formally, $cr_2(G) = \min cr(G_1) + cr(G_2)$, where $G_1 \cup G_2 = G$. Unfortunately, most of the results are counterexamples, which show that the study of biplanar crossing numbers is even harder than the study of crossing numbers.

One negative result to mention is refutation to Halton's conjecture (Halton 1991), which claims that every graph with maximum degree 6 or less is biplanar. Another negative result is that for any size exceeding a certain linear threshold, there exists a graph G of this size, which simultaneously has the following properties: $cr(G)$ is roughly as large as it can be for graphs of that size, and $cr_2(G)$ is as small as it can be for graphs of that size.

It is shown that $cr_2(G)$ for random graphs is roughly as large as it can be—although we cannot prove this property for any non-trivial fixed graph!

The positive results: We show that $cr_2(G) \leq (3/8)cr(G)$. Using this result recursively, we bound the thickness by $\Theta(G) = 2 + O(cr_2(G)^{.4057})$.

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