GANG OF FOUR REVISITED*

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Abstract: An explicit equilibrium is given for symmetric versions of the perturbed repeated prisoners' dilemma game of Kreps et al. (1982). The derivation is especially simple in a convenient special case. Analysis of the solution for each case shows that it is always possible to achieve full cooperation up to $O(\log(1/\delta))$ periods before the end of the horizon, where δ is the probability of the tit-for-tat type. This significantly improves on the result in Kreps et al.. It is also shown that no rational cooperation is possible unless the horizon is of at least this order. Journal of Economic Literature Classification Numbers: C73, D82.

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1. INTRODUCTION

One of the major events in the history of game theory was the demonstration in Kreps et al. (1982), that asymmetric information coupled with small probabilities of player irrationality can lead to significant cooperation in finitely repeated situations such as the finitely repeated prisoners' dilemma.

Unfortunately, Kreps et al. do not actually give an explicit equilibrium for their game, stating that, "[t]o present a sequential equilibrium in full detail for this game is difficult" because "the 'end play' of such equilibria are very complex." This is a problem because it makes the analysis of this game less transparent. It is also frustrating that, even in advanced game theory classes, for example, one cannot present a simple, explicit equilibrium to one of the most important games in the history of the field.[1]

Consider the following prisoners' dilemma game, with a, b, c > 0, and b > a (the zeros to the lower right are a harmless normalization).

	C	D
C	a, a	− c, b
D	b, − c	0, 0

Kreps et al. (1982) repeat this game a finite number of times, and allow for a small probability, δ, that one of the players is a "tit-for-tat" (TFT) type, i.e., a type which always plays tit-for-tat. Players know their own type, but not that of the other player. In this context, Kreps et al. prove that cooperation is possible until relatively near the end of the horizon. However, they do not present an explicit equilibrium.

The current paper presents a simple explicit equilibrium to this perturbed, repeated prisoners' dilemma game. Analysis of this equilibrium shows that cooperation is possible up to $O(\log(1/\delta))$ periods before the end of the horizon. This is significantly better than the bound of

$O(1/\delta)$ in Kreps et al. (see their step 8).

This result was also proven, for more general, asymmetric prisoners' dilemmas, in Theorem 1 of Conlon (2003). However, explicit equilibria were only constructed for the special case $b = a + c$. A related result was also proven in Neyman (1999), but with uncertainty about the length of the horizon, not about the types of the players. In addition, a result of this sort is suggested by the solution of the chain store game in Kreps and Wilson (1982).

We also show that the game must be repeated at least of order $\log(1/\delta)$ times to achieve any rational cooperation.[2] Again, this result follows from the proof of Theorem 1 in Conlon (2003), but the proof here is focused more directly on this result, and is therefore shorter.

The paper begins with an especially simple special case, where $b = a + c$ (see Assumption (5) below). It then turns to general symmetric prisoners' dilemmas.[3] Obtaining an equilibrium in the general case is computationally somewhat messier, but it involves similar ideas.

The first step in the construction is performed in the following section. Section 3 then presents the solution for the case $b = a + c$, while Sections 4 and 5 present the solutions for the cases $b < a + c$ and $b > a + c$, respectively (we always assume $2a > b - c$, as in Kreps et al. (1982)). Section 6 shows that cooperation by rational players is possible only if the total number of repetitions is at least of order $\log(1/\delta)$. Section 7 concludes.

2. THE FIRST STEP OF THE CONSTRUCTION

This section sets up the basic framework, and begins the construction. We start with a T-period repeated prisoners' dilemma, with payoffs illustrated above, and no discounting.

At this point, focus on symmetric equilibria in which, once rational players start to mix, they are willing to mix in every period but the last, unless someone actually defects, in which case both players defect from the next period on. Let V_t be the expected present value of the payoffs to a rational player, from period t to period T, inclusive, given that both players have cooperated previously. Let p_t be the probability which a player attaches to her opponent's cooperating in period t, given that both have cooperated previously. Note that p_t aggregates over both the rational and the TFT players.

In period T, rational players defect. Thus,

$$V_T = b\, p_T. \tag{1}$$

Consider, now, some period $t < T$, in which a rational player is willing to mix between cooperation and defection. Defection yields a payoff of $b\, p_t$, since both players will defect in all future periods. Cooperation in period t yields a payoff of $-c$ if the opponent defects, and a payoff of $a + V_{t+1}$ if the opponent cooperates, yielding an expected value of

$$(1 - p_t)(-c) + (a + V_{t+1})\, p_t.$$

Thus, if the player is willing to mix in period t, it must be the case that

$$V_t = b\, p_t = (1 - p_t)(-c) + (a + V_{t+1})\, p_t. \tag{2}$$

Solving the second equality in (2) for p_t gives

$$p_t = \frac{c}{a + c - b + V_{t+1}}. \tag{3}$$

Finally, if this player is also willing to mix or defect in period t + 1, then $V_{t+1} = b\, p_{t+1}$. Plugging this into (3) gives

$$p_t = \frac{c}{a + c - b + b\, p_{t+1}}. \tag{4}$$

If rational players are willing to mix between periods $T - K$ and $T - 1$, and willing to defect in period T, then the recursive formula (4) gives p_t for $T - K \le t \le T - 1$ as a function of p_T.

Equation (4) becomes especially simple under the following

ASSUMPTION: $b = a + c$. (5)

Under this assumption, (4) becomes

$$p_t = c/b\ p_{t+1} = A/p_{t+1}, \text{ so } p_t\, p_{t+1} = A, \tag{6}$$

where $A = c/b < 1$. In this case, p_{T-k} equals p_T for k even, and A/p_T for k odd, as long as $k \leq K$. Note that the recursion (6) works if and only if p_T is in the interval [A, 1].

Assumption (5) drastically simplifies the construction, and so, will be maintained in Section 3. Sections 4 and 5 will treat the more general case.

3. THE SECOND STEP FOR THE CASE b = a + c

Assume now that $b = a + c$ (this case was also considered in Conlon (2003), but is included here for completeness). Assume that, with prior probability δ, any given player is a TFT type, where a TFT type is a type that is constrained to always play tit-for-tat against the other player – that is, is constrained to play whatever the other player played last period. Let q_t be the posterior probability that a player is TFT, given that both players have cooperated in all periods strictly prior to t. Let $T - K$ be the first period in which rational players are willing to mix between cooperation and defection, and assume that all players mix or defect for all $t \geq T - K$. Thus, (6) holds for all t between $T - K$ and $T - 1$, inclusive.

Clearly, $q_t = \delta$ for $t \leq T - K$. Also, by Bayes' Rule,

$$q_t = q_{t-1}/p_{t-1}. \tag{7}$$

Thus, applying (7) repeatedly gives

$$q_t = \delta/p_{T-K}\, p_{T-K+1} \cdots p_{t-1}. \tag{8}$$

Also, given that both players have cooperated in all periods up to and including period $T - 1$, the only players who will cooperate in period T are the TFT types. Thus, $p_T = q_T$. It therefore follows that p_T must solve

$$p_T = \delta / p_{T-K}\, p_{T-K+1} \cdots p_{T-1}. \tag{9}$$

Choose K even such that

$$A \le A^{-K/2}\, \delta < 1. \tag{10}$$

There is always an even K which satisfies this inequality.

Since K is even, it follows that

$$p_{T-K}\, p_{T-K+1} \cdots p_{T-1} = (p_{T-K}\, p_{T-K+1}) \cdots (p_{T-2}\, p_{T-1}) = A^{K/2}, \tag{11}$$

where the last step follows from (6). Plugging this result into (9) gives

$$p_T = A^{-K/2}\, \delta. \tag{12}$$

By (10), this value of p_T lies in $[A, 1)$, so the recursion in (6) is valid. This therefore gives the equilibrium value of p_T, and so, the equilibrium value of p_t for $t \ge T - K$, by (6). For $t < T - K$, of course, $p_t = 1$.

Finally, the strategies of the rational players must be chosen to yield the aggregate probabilities p_t. Thus, let r_t be the equilibrium probability that a rational player cooperates in period t. Then $(1 - q_t)\, r_t + q_t = p_t$, so

$$r_t = (p_t - q_t)/(1 - q_t), \tag{13}$$

given previous cooperation by both players.

The probability r_t in (13) is nonnegative since $q_t \le p_t$. This last inequality may be seen as follows. First, since all of the p_t are bounded by one, the q_t are nondecreasing in t, by (7), and so, bounded by q_T. Thus, since $q_T = p_T < 1$, all of the q_t are less than one. Using (7) again shows that $q_t/p_t = q_{t+1} < 1$ for all $t \le T - 1$, so $q_t < p_t$ for all $t \le T - 1$. Finally, $q_T = p_T \le p_T$, so $q_t \le p_t$ for all t, so r_t really *is* nonnegative. In addition, from (13) it is clear that $r_t \le 1$. Also, since $q_T = p_T$, it follows that $r_T = 0$, as it should be, since rational players never cooperate in period T.

In summary, if the p_t are generated as in (6), given p_T, then rational players will be willing to mix as in (13). But if K and p_T are given in (10) and (12), then (9) is satisfied, and $p_T = q_T$. Thus, if the rational players follow the strategy profile, their expectations about the p_t will be rational, and their strategies will be rational given their expectations. The strategy profile is therefore a sequential equilibrium.

As a simple numerical example, let a = 3, b = 4, and c = 1. Also, let T = 8, and δ = 1/100. Then A = c/b = 1/4, so K = 6 and $p_T = p_8 = 64/100$, yielding

$$p_1 = 1,\quad p_2 = p_4 = p_6 = p_8 = 64/100,\quad \text{and}\quad p_3 = p_5 = p_7 = 100/256.$$

The q_t's and the r_t's are then given in Table 1. Thus, the equilibrium can be calculated fairly easily in this case. Finally, inspection of (10) proves the following proposition.

TABLE 1

t	q_t	r_t
1	1/100	1
2	1/100	63/99
3	1/64	24/63
4	4/100	60/96
5	1/16	21/60
6	16/100	48/84
7	1/4	9/48
8	64/100	0

PROPOSITION 1: In the case b = a + c, full cooperation is possible up to $O(\log(1/\delta))$ periods before the end of the horizon.

This is exactly what one would expect from Kreps and Wilson (1982), and generally much shorter than the bound obtained in Step 8 of Kreps et al. (1982) (roughly $O(1/\delta)$).[4] Thus, in the present case, the length of horizon needed to achieve cooperation is relatively short.

4. THE CASE $b < a + c$

For the case $b < a + c$, define the function

$$g(p) = \frac{c}{a + c - b + b\,p}. \tag{14}$$

Since $b < a + c$, this function maps the interval $[1 - (a/b), 1]$ into itself. This is clear since the function is decreasing in p, while

$$g(1-(a/b)) = 1 \in [1 - (a/b), 1], \text{ and } g(1) = c/(a + c) \in [1 - (a/b), 1]. \tag{15}$$

The last inclusion follows from $b < a + c$.

Equation (4) implies that

$$p_t = g(p_{t+1}). \tag{16}$$

Thus, as long as p_T is in $[1 - (a/b), 1]$, all the p_t's will be in this interval also. Thus, since $a < b$, these probabilities will be between zero and one, as probabilities should be.

Rewrite (16) as

$$p_{t-1} = \frac{c}{a + c - b + b\,p_t}. \tag{17}$$

Also, define x_t to be $1/p_t$, so (17) becomes

$$x_{t-1} = [(a + c - b)/c] + (b/c\, x_t) = A_1 + B_1/x_t, \tag{18}$$

where $A_1 = (a + c - b)/c$ and $B_1 = b/c$. Thus,

$$x_{t-1}\, x_t = A_1\, x_t + B_1. \tag{19}$$

Proceeding inductively, suppose

$$x_{t-k} \ldots x_{t-1}\, x_t = A_k\, x_t + B_k. \tag{20}$$

Then, lagging this back one period and multiplying by x_t gives

$$
\begin{aligned}
x_{t-k-1} \dots x_{t-1}\, x_t &= (x_{t-k-1} \dots x_{t-1})\, x_t = (A_k\, x_{t-1} + B_k)\, x_t = A_k\, x_{t-1}\, x_t + B_k\, x_t \\
&= A_k\,(A_1\, x_t + B_1) + B_k\, x_t = (A_k\, A_1 + B_k)\, x_t + A_k\, B_1 = A_{k+1}\, x_t + B_{k+1}.
\end{aligned} \tag{21}
$$

This gives a simple recursive formula for A_k and B_k, namely,

$$A_{k+1} = A_k\, A_1 + B_k, \text{ and } B_{k+1} = A_k\, B_1, \text{ where } A_1 = (a + c - b)/c \text{ and } B_1 = b/c. \tag{22}$$

Now, the condition $p_{T-K}\, p_{T-K+1} \dots p_T = \delta$, with $p_T \in [1 - (a/b), 1]$, becomes

$$A_K/p_T + B_K = 1/\delta, \text{ for some K and some } p_T \in [1 - (a/b), 1]. \tag{23}$$

Thus, if (23) has a solution in K and p_T, then p_T is given by

$$p_T = \delta\, A_K/(1 - \delta\, B_K). \tag{24}$$

Thus, once K, the A_K's, and the B_K's are calculated, it becomes trivial to calculate p_T. Then the recursion in (17) gives the p_t's for $T - K \leq t < T$. Finally, the rational players' mixed strategy probabilities of cooperating are given by (13), as before. Thus, a solution to (23) yields an equilibrium for the game. The following lemma shows that such a solution always exists.

LEMMA 1: Problem (23) has a solution in K and p_T.

PROOF: See Appendix A.

Thus, the approach in Section 3 can be generalized to the case where $b < a + c$. This solution also allows us to estimate the length of horizon needed to achieve full cooperation:

PROPOSITION 2: In the case $b < a + c$, full cooperation is possible up to $O(\log(1/\delta))$ periods before the end of the horizon.

PROOF: The proof of Lemma 1 also shows that the A_k and B_k grow exponentially, at a rate $\lambda_1 > 1$. Thus, K in (23) must be of the same order of magnitude as the log, base λ_1, of $1/\delta$.

5. THE CASE b > a + c

The case b > a + c is very similar to the previous case. There is one major difference, however: if b > a + c, then the function g(p) does not map [1 − (a/b), 1] into itself. Thus, the procedure above may yield values for p_t which are either negative or larger than one. The solution to this problem is to simply turn the procedure around.

The recursive relation in (4) can be inverted to obtain

$$p_{t+1} = ([b - a - c]/b) + (c/b\, p_t) = C_1 + D_1/p_t, \tag{25}$$

where $C_1 = (b - a - c)/b$ and $D_1 = c/b$. Thus, letting

$$h(p) = ([b - a - c]/b) + (c/b\, p) \tag{26}$$

gives

$$p_{t+1} = h(p_t). \tag{27}$$

The function h(p) maps the interval [c/(a + c), 1] into itself. Thus, if T − K is the first period in which rational players are willing to mix, then, as long as $p_{T-K} \in [c/(a + c), 1]$, (27) will yield a well defined sequence of p_t's for $t \geq T - K$.

Next, note that multiplying (25) by p_t gives

$$p_t\, p_{t+1} = C_1\, p_t + D_1. \tag{28}$$

Proceeding inductively, suppose

$$p_t\, p_{t+1} \cdots p_{t+k} = C_k\, p_t + D_k. \tag{29}$$

Lagging this forward one period and multiplying by p_t gives

$$\begin{aligned} p_t\, p_{t+1} \cdots p_{t+k+1} &= p_t\,(p_{t+1} \cdots p_{t+k+1}) = p_t\,(C_k\, p_{t+1} + D_k) = C_k\, p_t\, p_{t+1} + D_k\, p_t \\ &= C_k\,(C_1\, p_t + D_1) + D_k\, p_t = (C_k\, C_1 + D_k)\, p_t + C_k\, D_1 = C_{k+1}\, p_t + D_{k+1} \end{aligned} \tag{30}$$

This gives a simple recursive formula for C_k and D_k, namely,

$$C_{k+1} = C_k C_1 + D_k, \text{ and } D_{k+1} = C_k D_1, \text{ where } C_1 = (b - a - c)/b \text{ and } D_1 = c/b. \tag{31}$$

Now, the condition $p_{T-K}\, p_{T-K+1} \ldots p_T = \delta$, with $p_{T-K} \in [c/(a+c), 1]$ becomes

$$C_K\, p_{T-K} + D_K = \delta, \text{ for some K and some } p_{T-K} \in [c/(a+c), 1]. \tag{32}$$

Thus, if (32) has a solution in K and p_{T-K}, then p_{T-K} is given by

$$p_{T-K} = (\delta/C_K) - (D_K/C_K). \tag{33}$$

Once K, the C_K's, and the D_K's, are calculated, it is then straightforward to calculate p_{T-K}. Then the recursion in (25) yields the p_t's for $T - K < t \leq T$. Finally, the rational players' mixed strategy probability of cooperating in period t will be given by (13). Thus, a solution to (32) leads directly to an equilibrium for the game. The following lemma shows that such a solution always exists.

LEMMA 2: Problem (32) has a solution in K and p_{T-K}.

PROOF: See Appendix B.

Thus, the approach in Section 3 can also be generalized to the case $b > a + c$ with some minor modifications. Finally, a proof similar to that of Proposition 2 above shows:

PROPOSITION 3: In the case $b > a + c$, full cooperation is possible up to $O(\log(1/\delta))$ periods before the end of the horizon.

6. THE HORIZON CANNOT BE TOO SHORT

Propositions 1, 2, and 3 established for the three cases that full cooperation is possible up to $O(\log(1/\delta))$ periods before the end of the horizon. Proposition 4 now shows that this is the best result possible in the present framework.

PROPOSITION 4: The number of periods needed to obtain some cooperation by rational players in the present framework is at least order of $\log(1/\delta)$.

PROOF: See Appendix C.

Note that Propositions 1, 2, and 3 show that full cooperation is possible with horizons of $O(\log(1/\delta))$, while Proposition 4 shows that least this order of periods is needed for any rational cooperation to be possible. The result in Proposition 4 also follows from the regularity of equilibria established in the proof of Theorem 1 in Conlon (2003).

Proposition 4 also implies that repeated prisoners' dilemmas are somewhat different from centipede games. In centipede games, some rational cooperation is possible, even with three stages and δ arbitrarily close to zero, whereas Proposition 4 shows that *no* rational cooperation is possible unless the horizon is relatively long.

7. CONCLUSION

This paper has constructed a simple explicit solution to the perturbed finitely repeated prisoners' dilemma, under Assumption (5). This solution makes it easy to present an explicit equilibrium to one of the most important games in the history of the field. The above equilibrium may, in fact, be considerably easier to derive and understand than the equilibrium for the chain store game in Kreps and Wilson (1982) (see footnote 1).

The later sections extended the construction to arbitrary symmetric prisoner's dilemmas. Unfortunately, the solutions to the key problems, (23) and (32), are somewhat more difficult. Also, the proof that a solution exists is simple in Section 3 but nontrivial in Sections 4 and 5. However, if one accepts the existence of solutions to (23) and (32), then the constructions in Sections 4 and 5 are conceptually almost as simple as the construction in Section 3.

The paper has also shown that cooperation is possible in the perturbed repeated prisoners' dilemma up to $O(\log(1/\delta))$ periods before the end of the horizon, and that this is best possible. This significantly improves the bound in Kreps et al. (1982).

APPENDIX A: PROOF OF LEMMA 1

We begin by using standard techniques to solve the system of difference equations in (22). First, note that, if A_k and B_k satisfy (22), then A_k satisfies the difference equation

$$A_{k+2} - A_1 A_{k+1} - B_1 A_k = 0. \qquad \text{(A1)}$$

This generates the characteristic equation

$$f(\lambda) = \lambda^2 - A_1 \lambda - B_1 = 0, \qquad \text{(A2)}$$

with solutions

$$\lambda_1 = [A_1 + (A_1^2 + 4 B_1)^{1/2}]/2, \qquad \text{(A3)}$$

and

$$\lambda_2 = [A_1 - (A_1^2 + 4 B_1)^{1/2}]/2. \qquad \text{(A4)}$$

Thus, the expressions

$$A_k = \lambda_1^k + \lambda_2^k \text{ and } B_k = (b/2\,c)[\lambda_1^{k-1} + \lambda_2^{k-1}] \qquad \text{(A5)}$$

solve the difference equations (22). Furthermore, they satisfy the initial conditions when $k = 1$ (see the second half of (22)). Thus, A_k and B_k are given by (A5).

We now determine the signs and magnitudes of λ_1 and λ_2. First, λ_1 is clearly positive and λ_2 is clearly negative. Also, λ_2 is less than λ_1 in absolute value, so as k gets large, the behavior of λ_1^k determines the behavior of A_k and B_k. Finally, $\lambda_1 > 1$. This follows since

$$f(1) = 1 - A_1 - B_1 = 1 - [(a + c - b)/c] - (b/c) = - a/c < 0, \qquad \text{(A6)}$$

while $f(\lambda)$ is positive for λ large. Thus, $f(\lambda) = 0$ must have a solution bigger than one. Since λ_1 is the bigger of the two solutions, it must be bigger than one.

This implies that A_k and B_k grow exponentially in k. We next use this to show that, for any δ, there is a K and a p_T satisfying (23), so $x_T = 1/p_T \in [1, b/(b - a)]$ and $A_K x_T + B_K = 1/\delta$.

Since A_k and B_k grow exponentially, $A_K (b/[b - a]) + B_K$ will eventually be bigger than $1/\delta$. Let K be the smallest number such that

$$A_K (b/[b - a]) + B_K \geq 1/\delta. \tag{A7}$$

Then $A_{K-1} (b/[b - a]) + B_{K-1} < 1/\delta$. We want to show that

$$A_K + B_K < 1/\delta. \tag{A8}$$

Inequalities (A7) and (A8) will then show that there is an $x_T \in [1, b/(b - a)]$ with $A_K x_T + B_K = 1/\delta$, as is to be proven. To prove (A8), note that

$$\begin{aligned} A_K + B_K &= (A_{K-1} A_1 + B_{K-1}) + A_{K-1} B_1 \\ &= A_{K-1} (A_1 + B_1) + B_{K-1} = A_{K-1} ([a + c]/c) + B_{K-1}. \end{aligned} \tag{A9}$$

Now, since $b < a + c$, it follows by cross multiplying that $[a + c]/b < b/[b - a]$. Thus, the previous equation gives

$$A_K + B_K \leq A_{K-1} (b/[b - a]) + B_{K-1} < 1/\delta. \tag{A10}$$

This completes the proof.

APPENDIX B: PROOF OF LEMMA 2

We begin by using standard techniques to solve the system of difference equations in (31). First, note that, if C_k and D_k satisfy (31), then C_k satisfies the difference equation

$$C_{k+2} - C_1 C_{k+1} - D_1 C_k = 0. \tag{B1}$$

This generates the characteristic equation

$$f(\lambda) = \lambda^2 - C_1 \lambda - D_1 = 0 \tag{B2}$$

(note that this function f differs from the f in Appendix A). Equation (B2) has solutions

$$\lambda_1 = [C_1 + (C_1^2 + 4 D_1)^{1/2}]/2, \tag{B3}$$

and

$$\lambda_2 = [C_1 - (C_1^2 + 4 D_1)^{1/2}]/2. \tag{B4}$$

Thus, the expressions

$$C_k = \lambda_1^k + \lambda_2^k \text{ and } D_k = (c/2\, b)[\lambda_1^{k-1} + \lambda_2^{k-1}] \tag{B5}$$

solve the difference equations (31). Furthermore, they satisfy the initial conditions when $k = 1$ (see the second half of (31)). Thus, C_k and D_k are given by (B5).

We now determine the signs and magnitudes of λ_1 and λ_2. First, λ_1 is clearly positive and λ_2 is clearly negative. Also, λ_2 is less than λ_1 in absolute value, so as k gets large, the behavior of λ_1^k determines the behavior of A_k and B_k. Finally, $\lambda_1 < 1$. This follows since

$$f(1) = 1 - C_1 - D_1 = 1 - [(b - a - c)/b] - (c/b) = a/b > 0, \tag{B6}$$

while $f(0) = - D_1 < 0$. Thus, $f(\lambda) = 0$ must have a solution between zero and one. Since $\lambda_2 < 0$, λ_1 must be the solution between zero and one, so $\lambda_1 < 1$.

Since $1 > \lambda_1 > |\lambda_2|$, C_k and D_k decline roughly exponentially towards zero as k grows. We now use this to show that, for any δ, there is a K and a p_{T-K} such that (32) is met, i.e., such that $p_{T-K} \in [c/(a + c), 1]$ and $C_K\, p_{T-K} + D_K = \delta$.

Since C_k and D_k decline exponentially, $C_K (c/[a + c]) + D_K$ will eventually be smaller than δ. Let K be the smallest number such that

$$C_K (c/[a + c]) + D_K \leq \delta. \tag{B7}$$

Then C_{K-1} $(c/[a + c]) + D_{K-1} > \delta$. We want to show that

$$C_K + D_K > \delta. \tag{B8}$$

Inequalities (B7) and (B8) will then show that there is a $p_{T-K} \in [c/(a + c), 1]$ with $C_K\, p_{T-K} + D_K = \delta$, as is to be proven. To prove (B8), note that

$$\begin{aligned} C_K + D_K &= (C_{K-1}\, C_1 + D_{K-1}) + C_{K-1}\, D_1 \\ &= C_{K-1}\, (C_1 + D_1) + D_{K-1} = C_{K-1}\, ([b - a]/b) + D_{K-1}. \end{aligned} \tag{B9}$$

Now, since $b > a + c$, it follows by cross multiplying that $[b - a]/b > c/[a + c]$. Thus, the previous equation gives

$$C_K + D_K \geq C_{K-1}\, (c/[a + c]) + D_{K-1} > \delta. \tag{B10}$$

This completes the proof.

APPENDIX C: PROOF OF PROPOSITION 4

Note first that it is no longer enough to consider symmetric equilibria. Thus, we allow the equilibrium to be asymmetric. Let V_t^I and V_t^{II} be the expected equilibrium payoffs to Players I and II, respectively, from periods t through T, inclusive, let p_t^{II} be the probability that Player I assigns to Player II cooperating in period t, given that both players cooperated previously, and let p_t^I be the corresponding probability for Player I cooperating.

It is possible that one or both of rational Player I and rational Player II will refuse, with probability one, to cooperate after some $T' \leq T - 1$, even if both cooperated previously. Assume without loss of generality that Player I is willing to mix or cooperate until T', and that Player II is <u>not</u> willing to mix after T' (Player II may not even mix in period T').

To obtain a lower bound on T', we need a lower bound for p_t^{II} such that Player I would be willing to cooperate. First, if Player I mixes or defects in period t + 1, then $V_{t+1}^I = p_{t+1}^{II}\, b \leq b$.

Next, suppose that, in equilibrium, Player I cooperates with probability <u>one</u> for periods t + 1 through t + k, and mixes or defects in period t + k + 1. Then

$$V_{t+1}^{I} \leq k\,a + b. \tag{C1}$$

Note that this inequality also holds for k = 0 (i.e., if Player I mixes or defects in t + 1, $V_{t+1}^{I} \leq b$). For Player I to be willing to cooperate with positive probability in period t, it must be the case that

$$p_t^{II}[a + V_{t+1}^{I}] - (1 - p_t^{II})\,c \geq b\,p_t^{II}, \tag{C2}$$

or, using (C1),

$$p_t^{II}\,[(k + 1)\,a + c] \geq p_t^{II}\,[a + c - b + V_{t+1}^{I}] \geq c. \tag{C3}$$

Here the first inequality comes from (C1), and the second inequality comes from (C2). Rearranging (C3) gives

$$p_t^{II} \geq \frac{c}{c + (k + 1)\,a} = \frac{1}{1 + (k + 1)(a/c)}. \tag{C4}$$

If k = 0, this yields $p_t^{II} \geq 1/(1 + (a/c))$. If k = 1, (C4) yields $p_t^{II} \geq 1/(1 + 2(a/c))$ and (C4) shifted forward one period yields $p_{t+1}^{II} \geq 1/(1 + (a/c))$. If k ≥ 2, then, since Player I cooperates with probability one up to period t + k, Player II will cooperate up to at least t + k − 1. Thus,

$$p_t^{II}\,p_{t+1}^{II} \cdots p_{t+k}^{II} = p_t^{II}\,p_{t+k}^{II} \geq \frac{1}{1 + (k + 1)(a/c)}\;\frac{1}{1 + (a/c)}, \tag{C5}$$

where the last step follows from (C4). Note that (C5) also encompasses the case k = 1, but not the case k = 0. Note also that (C5) holds even if rational types of Player II have defected for sure before period t. In that case, previous cooperation implies that Player II must be a TFT type. Thus, the left hand side of (C5) will be one, so (C5) will hold trivially.

Using $p_t^{II} \geq 1/(1 + (a/c)) \geq 1/[1 + (a/c)]^2$ for k = 0 and (C5) for k ≥ 1 gives

$$p_t^{II} p_{t+1}^{II} \cdots p_{t+k}^{II} \geq 1/[1 + (a/c)]^{2k+2} = 1/C^{k+1} \tag{C6}$$

for all $k \geq 0$, where

$$C = [1 + (a/c)]^2 > 1. \tag{C7}$$

Here, for $k \geq 1$, we have used

$$\begin{aligned}[1 + (k+1)(a/c)][1 + (a/c)] &\leq [1 + (a/c)]^{k+1}[1 + (a/c)] \\ &= [1 + (a/c)]^{k+2} \leq [1 + (a/c)]^{2k+2} = C^{k+1}.\end{aligned} \tag{C8}$$

Thus, we can break the T'-period time interval from 1 to T' into subintervals of length k + 1 for various values of k, such that the sum of all the k + 1's equals T', and such that the products of the p_t^{II}'s within each subinterval satisfy (C6).

A simple analogue to (9) (with rational Player II never cooperating after period T'), next gives

$$p_1^{II} p_2^{II} \cdots p_{T'}^{II} p_{T'+1}^{II} = \delta. \tag{C9}$$

This gives $(1/C^{T'})\, p_{T'+1}^{II} \leq \delta$.

Now, since Player I is willing to mix or cooperate in period T', $p_{T'+1}^{II} \geq (b - a)/b$. This follows from (4), with p_t and p_{t+1} being p_t^{II} and p_{t+1}^{II}. For Player I to be willing to mix or cooperate, the right hand side of (4) should be less than or equal to one, which implies that $p_{t+1}^{II} \geq (b - a)/b$.

Thus, $C^{T'} \geq (1/\delta)[(b - a)/b]$ with $C > 1$ from (C7). It follows that T', and so T, must be at least of the order $\log(1/\delta)$ as δ gets small, for some rational cooperation to be achievable in this framework.

NOTES

1. Textbooks that treat the related chain store game of Kreps and Wilson (1982) include Fudenberg and Tirole (1991), Ordeshook (1986), and Osborne and Rubinstein (1994) (see also problem 6.4 in Eichberger (1993)). However, chain store games are more complicated than repeated prisoners' dilemmas, since, in repeated prisoners' dilemmas, there is only one type of history with an interesting continuation, i.e., the history in which everyone cooperates up to the present. By contrast, chain store games have many possible histories with interesting continuations. All histories in which the incumbent has not yet accommodated are interesting, and these histories can vary according to which potential entrants have decided to enter.

Gibbons (1992) treats an example of the repeated prisoners' dilemma in which the probability of tit-for-tat behavior is so high that defection occurs only in the last two periods. Finally, Kreps (1990) and Myerson (1991) consider numerical examples of the closely related perturbed centipede game, and Tirole (1988) solves a reputation-for-quality game which is almost identical to the centipede game. Perturbed centipede games are similar to the Kreps et al. (1982) perturbed repeated prisoners' dilemma game treated here, though only one player moves at a time in the centipede game while players move simultaneously in the prisoners' dilemma game. This implies somewhat different behavior (see Section 6).

Tirole (1988, pp. 123-26, 128-29) briefly considers a numerical example of the Kreps et al. (1982) game and states (p. 259, Footnote 38) that "the equilibrium [for this game] and its derivation are formally similar" to the equilibrium in the centipede-like reputation-for-quality game mentioned above. Tirole's numerical example satisfies Assumption (5) below, and thus falls under our simple special case. Tirole, however, does not actually derive the equilibrium for this game.

2. Small irrationalities thus yield less cooperation in repeated discrete action games such as prisoners' dilemmas than in repeated continuous action games such as Cournot games, where full cooperation is generally possible with much shorter horizons, of order O(loglog(1/δ)) (see

Conlon (1996)).

3. Note that Kreps et al. (1982) assume that only one of the players can be a tit-for-tat type, while we assume that both can be. In addition, this paper only constructs a single equilibrium along which cooperation occurs, while Kreps et al. prove that cooperation occurs in all equilibria of sufficiently long finitely repeated prisoners' dilemmas perturbed by TFT types.

However, the existence of equilibria involving cooperation may be the most interesting result in Kreps et al. (1982). This is in part because it is the possibility of cooperation that is surprising here, and in part because the stronger result, that cooperation always occurs under TFT perturbations, is not robust to slight variations in the assumption that all perturbations are TFT.

4. The formula in Step 8 of Kreps et al. (1982), modified for the current framework, is $3 + (2b + 4c)/a\delta$. For the numerical example above, this indicates that full cooperation is possible 403 periods before the end of the horizon. By contrast, the above solution achieves full cooperation seven periods before the end of the horizon. For smaller values of δ, of course, the difference would be even more spectacular. On the other hand, Kreps et al. assume one sided uncertainty while we assume two sided uncertainty. This, however, only makes a difference of roughly one period.

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