Raymond D. Mindlin and Applied Mechanics

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R. D. Mindlin and Applied Mechanics

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Abstract
Professor Raymond D. Mindlin (1906-1987), member of U.S. National Academy of Engineering and of National Academy of Science, devoted his entire professional life to engineering education at Columbia University in the City of New York. His lifetime research contributions to the science of Mechanics and applications to civil, mechanical, electrical and ultrasonic engineering are briefly discussed in this lecture. The major contributions may be grouped under nine subjects: Three-Dimensional Theory of Elasticity, Photoelasticity and Photoviscoelasticity, Generalized Elastic Continua, Contact Stresses and Mechanics of Granular Media, Waves and Vibrations in Isotropic and Anisotropic Plates, Vibrations and Wave Propagation in Rods, Piezoelectric Crystals and Electro-Elasticity, Lattice Theories and Continuum Mechanics, and Elastic Waves in a Parallelepiped. The first eight subjects had been summarized by Mindlin’s former students in eight articles which are compiled and edited by G. Herrmann in “R. D. Mindlin and Applied Mechanics” (Pergamon Press, New York, 1974), and the last named subject was a continuation of his early work on vibrations of elastic plates after retirement.
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Raymond D. Mindlin-His Life (1906-1987)

Born in New York City, New York, in 1906; educated in private elementary and preparatory schools.

College Education at Columbia University in the City of New York (1924-36): Bachelor of Arts in 1928 (B.A.); Bachelor of Science in 1931 (B.S.); Civil Engineer in 1932 (C.E.); Doctor of Philosophy in 1936 (Ph.D.).

Teaching and Research Career in the School of Civil Engineering, Columbia University (1933-1975):
- Assistant (1933-38);
- Instructor (1938-39);
- Assistant Professor (1940-44);
- Associate Professor (1945-46);
- Professor (1947-67);
- James Kip Finch Professor of Applied Science (1967-75);
- Retired in 1975.

World War II Service (1942-46) at the Johns Hopkins University Applied Physics Laboratory:
- Awarded the Presidential Medal for Merit in 1946 by US Navy (for developing the proximity fuse of the anti-aircraft gun shell).

Research Contributions: to the fields of applied mechanics, structural engineering, geotechnical engineering, applied physics (acoustics, photoelasticity, piezoelectricity).

Honors and Awards: member of the National Academy of Engineering (1966), member of the National Academy of Sciences (1973),
- von Karman Medal of the ASCE in 1961,
- Timoshenko Medal of the ASME in 1964,
- C.B. Sawyer Award of the Army Electronics Command in 1967,
- National Medal of Science (a Presidential Award) in 1979.

Died in Hanover, New Hampshire in 1987
Mindlin's Publications (1934-1950)

4. Force at a point in the interior of a semi-infinite solid, Physics, 7, 166–169 (1936).
11. The quadrangular strain-contour, Civil Engineering, 5, 88–91 (1938).
17. Stress system in finite rectangular plates with corner stresses (with D. C. Derrig), Journal of Applied Physics, 10, 734–735 (1939).
22. The analogy between multiply-connected sheets and disks, Quarterly of Applied Mathematics, IV, 376–390 (1940).

*Not part of this collection.
Mindlin’s Publications (1950-1962)


[38] Mindlin’s Publications (1950-1962)


Mindlin’s Publications (1962-1972)


Mindlin’s Publications (1962-1972)


Mindlin's Publications (1972-1986)


*[117] Not part of this collection.*

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*Surrounded by well-wishers observing his 80th birthday and marking his forthcoming mandatory retirement from Columbia (Boulder, Colorado, 1974).*


Remarks on the Past, Present, and Likely Future of Photoelasticity

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Abstract—The Applied Mechanics Review summary paper by Mindlin is taken as the point of departure for a review of the past and present status of photoelasticity and a projection of the future. Some later developments are described and supplemental remarks made on earlier work. Recent advances in optical techniques and in optical analysis are seen as opening the way to many more new and valuable experimental and theoretical studies. Far more sophisticated modeling of materials and the more extensive use of photoelasticity as a tool of materials science and engineering appear as likely consequences.

1 Introduction

The optical techniques of experimental mechanics have enormous appeal beyond their intrinsic intellectual content and their utility in practice. Over the past fifty years photoelasticity, holography, and related methods provided one of the most useful experimental methods in research work and practicing engineer alike. High as this utility has been, the continued dominance today of optically oriented papers in the technical literature probably is more a result of esoteric satisfaction and of the host of fascinating observations which challenge the researcher to provide explanations.

In the last few years finite element methods have taken over many of the utilities functions which photoelasticity used to serve. Simultaneously, however, fundamental advances in our understanding of materials, and the desire for further advances, have opened new areas for photoelastic research. As a result one can foresee for the coming years profound changes in the directions of research and the areas of application of photoelasticity.

There will be greater understanding and use made of the meaning of the birefringence which is produced in a wide variety of materials. Attention will be directed to large deformations with both large strain and large rotation in the visco-elastic-plastic domain. Birefringence of material, as a material property, will be given far more attention in living as well as in inanimate fluids and solids. Crystals and polycrystals will be studied more for their own sake.
Development of the Three-Dimensional Theory of Elasticity

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1 Introduction

The theory of elasticity has for its objective the determination of the state of stress induced by load as borne by an ideal elastic material. Such a determination is a prerequisite for progress in the study of a wide variety of technological and scientific questions, ranging from structural engineering to the assessment of geophysical hypotheses. It is not surprising, therefore, that the subject developed relatively early in the history of continuum mechanics. By the close of the nineteenth century it had assumed a form recognizable today. This development at the hands of Cauchy, Navier, de St. Venant, F. Neumann, Kirchhoff, and Kelvin, among many distinguished authors, has been described in the treatise of Todhunter and Pearson [1] and has been comprehensively reviewed by Truesdell [2]. It is understandable also that a brief account can no longer do justice to the entire theory of elasticity. Fortunately, a number of eminent works develop the subject at length. Among these the books by Green and Zerna [3], Luir's [4], Sokolikoff [5], and Novozhilov [6] have been influential. Excellent accounts by Trefftz [7], Stuecklen, and Barry [8], Truesdell and Toupin [9], and Curtiss [10] appear in successive editions of the Handbuck der Physik. The treatises of Leder [11], Timoshenko and Goodier [12], Blot [13], and Solomon [14] contain detailed descriptions of many particular problems.

Attention is here confined to a description of leading developments in the linear three-dimensional theory of continuum elastomechanics. This excludes such important areas as finite deformations and structural stability, as well as the architectonic synthesis of two-dimensional elastostatics associated with the names of Kolosov and Muskhelishvili. The limitation is intended to make it possible to trace, in brief space, several unifying lines of twentieth-century activity in three-dimensional elasticity: the development of general methods of analysis, interaction with other field theories, and embedding in an axiomatically rigorous field theory of continuum mechanics.

These topics are developed by Green and Adkins [15], Truesdell and Neil [16], Gager [17], and England [18], these also.
Introduction

During the 1940s, R. L. Wegel at the Bell Telephone Laboratories in Murray Hill was studying the propagation of stress waves in granular aggregates. His interest in the subject was motivated by the pressing demand in the telephone industry for a rational method of designing the cation microphones, to the present day the most widely used type. The essential component of this device is an assembly of packed, electrically conducting carbon granules in which variation of acoustic pressure, caused by means of a diaphragm, creates changes in contact area between the individual grains. These changes, in turn, alter the electrical resistance of the system and hence provide variations in current flow.

Such analyses as existed were all based on the assumption that relative displacement of the grains occurred in the direction normal to the contact. Wegel was able to demonstrate, by means of experiments on pairs of rubber balls that the tangential compliance of the contact was of the same order of magnitude as its normal compliance and argued that it must therefore be taken into account in any analysis of the aggregate.

Sometimes in 1946 or 1947 R. D. Mindlin then serving as a consultant to the Laboratories, visited Wegel, was shown the results of the experiments and was urged to look into the question of predicting the tangential compliance mathematically. The basis of Mindlin's initial investigation in this field [262] was the germinal paper in a long series on frictional contact, by himself and his students, and led to a number of subsequent studies on the statical and dynamical response of granular arrays.

The aim of this article is twofold: to present the results of recent work in contact theory and to discuss the application of these results in the construction of a theory of mechanics of granular materials. Accordingly, the subject

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\[\text{Numbers preceded by the letter M in brackets refer to the Publications of R. D. Mindlin listed after the Preface to this volume.}\]

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Waves and Vibrations in Isotropic and Anisotropic Plates

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1 Introduction

Since Chladni demonstrated the nodal patterns of a vibrating plate in 1787, the subject of vibration of plates has fascinated many scientists and engineers. In a prize-winning memoir, offered for the third time by the Academy of Science of the French Institute in 1815, Sophie Germain [1] obtained the equation for an isotropic plate in bending motion. At about the same time, the subject also engaged the interest of many distinguished scholars including Lagrange, Legendre, Poisson, Cauchy, and Euler. The equation which is listed as Eq. (2.1) in this article is now referred to as the Germain-Lagrange equation because of certain contributions which Lagrange made when the paper was submitted for the prize.

The publication of Germain's memoir also started a prolonged controversy on the proper boundary conditions of a thin plate in bending. Saint-Venant, Thomas, Tait, Mathieu, Boussinesq, and Kirchhoff all made contributions to this subject. The paper by Kirchhoff [2] merits particular attention as it contained a complete derivation of the plate equation and the correct boundary conditions based on a variational method. It also contained a theorem of the uniqueness of the solutions and a comparison of theoretical results with Chladni's experiments.

At the same period of time, the in-plane motion of a plate was also investigated. The governing equations for a thin isotropic plate in extensional motion were derived by Poisson [3], and they are listed as Eq. (2.3). However, in contrast to flexural motion, the subject of extensional vibrations of a thin plate attracted much less attention until very recently.

In 1880, Rayleigh [4] published a paper which marked a turning point in the history of the theory of plates. In this paper, Rayleigh determined the vibrations of an homogeneous, isotropic plate of arbitrary thickness from the
Vibrations and Wave Propagation in Rods

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Abstract—This article devoted to vibrations and wave propagation in rods does not attempt to cover all aspects of the subject. The enormous amount of research on the dynamic behavior of rods and the extensive literature that has resulted has forestalled a selection of the topics that are described. In deciding what to omit and what to include the guide has been Professor Mindlin’s own research. Included are those topics on which he himself worked or which he directly influenced.

The article describes theoretical studies directed to interstrip, linearly elastic rods. Space has been assigned to the various topics roughly in proportion to the attention each has received in the literature. The first section, which is devoted to extensional waves in rods, accordingly, is the longest. Torsional vibrations of rods is treated briefly and flexural vibrations in detail. The section on other vibrations in circular rods is short, after which vibrations in rods of non-circular section discussed at greater length. In the final section, research is described in which Professor Mindlin was not directly involved but in which his theories were instrumental in the solution of the problem.

Introduction

Vibrations and wave propagation in rods have long been attractive subjects for scientific and engineering research. Interest began over two hundred years ago with Daniel Bernoulli, and since this beginning the tempo of research has increased, reaching its peak in the years since the Second World War. It is possible to speculate on the reasons why the dynamics of rods has aroused so much attention. Of all bounded bodies of arbitrary length, the rod has the simplest enclosing boundary. It is a possible mechanism for delay lines and wave guides. Furthermore, wave propagation in rods has been studied very recently to identify material properties.

The enormous amount of research on the dynamic behavior of rods and the extensive literature that has resulted, however, makes it difficult to summarize the research in a single article. Thus it was not possible to include all

Piezoelectric Crystals and Electro-elasticity

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Abstract—The classical, linear theory of piezoelectricity is briefly reviewed. Applications to piezoelectric vibrations are discussed for plates of indeterminate form, of uniform and variable thickness, with full and partial coatings, and under initial stress.

Recently developed theories of elastic disturbances, i.e., the polarization gradient theory and the theory of dislocation dislocations, are presented. Their accomplishments in many physical phenomena not accounted for by the classical theory and their convergence and relations to the long-wave limits of dynamical theories of crystal lattices are discussed.

Introduction

Since the development of the piezoelectric resonator by Curie in 1918, oscillators and filter circuits have relied heavily on piezoelectric crystals for ultra-precision requirements in frequency standards and for frequency control. Bars, cylinders, and especially plates, cut from the crystals and coated with thin metallic electrodes, are the elements most often used in such applications. They operate at mechanical resonance with an alternating voltage of high frequency imposed on the electrodes. The classical, linear theory of piezoelectricity is sufficient for the analysis in most instances. In this article, we will first briefly review this basic theory and consider its application to the study of plate vibrations.

Quartz is probably the most widely used crystal [1]. Since the electromechanical coupling, i.e., the piezoelectric effect, is weak in quartz, the analysis of quartz plate vibrations can be simplified for many applications by considering only the elastic field. Hence, frequent reference will be made to studies of quartz in which the piezoelectric effect is neglected.

In the remaining portion of the article, some recent developments in electro-elasticity are introduced. It is known that the differential equations of the classical piezoelectricity are not the long-wave, low-frequency limit of
Theories of Plates by R. D. Mindlin

Germain-Lagrange's Equation of Plates (1803-1812)

\[ V^2(DW^w) = \rho \frac{\partial^2 W^w}{\partial t^2} - q(x,y,t) \]

*Transverse Motion of a Plate in xy plane

- plate thickness, \( h \), mass density

\[ D = \frac{Eh}{3(1-\nu^2)}, \quad f = \frac{h^2}{12} \]

- Flexural Rigidity

- Young's modulus

- shear modulus

- Poisson's ratio

Mindlin's Equation of Plates (1951)

\[ (\nabla^2 + \frac{\kappa^2}{c^2})^2 \Delta w + \rho \frac{\partial^2 w}{\partial t^2} = q(x,y,t) \]

\( c^2 \) - shear coefficient

Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates

R. D. MINDLIN

A two-dimensional theory of thermal motions of isotropic, elastic plates is presented, which allows for the inclusion of effects of rotatory inertia and shear. The theory is based on a formulation of the governing equations, and it includes the effects of thermal expansion and contraction. The equations are derived from the general equations of linear elasticity, and they are applicable to a wide range of plate thicknesses, from thin to thick.

\[ D = \frac{Eh}{1-\nu^2}, \quad G = \frac{E}{2(1+\nu)}, \quad \kappa = \sqrt{\frac{E}{2(1-\nu^2)}} \]

* "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates" in Journal of Applied Mechanics, 1951
Eqs. of Elasticity and Eqs. of Plate

1. Stress-Equation of Motion
\[ \sigma_{ij} + \partial_\alpha \sigma_{\alpha j} + \partial_\beta \sigma_{i \beta} = \rho \ddot{u}_i \]

2. Stress-Displacement Relation
\[ \sigma_{ij} = \frac{1}{2} \left( \varepsilon_{ij} + \varepsilon_{ji} \right) \]

3. Displacement- Equation of Motion (Navier-Cauchy, 1822-28)
\[ (\lambda + 2\mu)\partial_\alpha \varepsilon_{ij} + \mu \partial_\alpha \partial_\beta \varepsilon_{i \beta} + \rho \ddot{u}_i = 0 \]

Plate displacement
\[ u_i (x, t) = \omega \varphi_i (x, y, t) \]
\[ u_i (x, t) = \pi (x, y, t) \]

Plate stress
\[ [Q_i, Q_j] = \int \frac{1}{2} \left( \sigma_{ij} + \sigma_{ji} \right) dx \]

1. Plate Stress Equation of Motion, \( \ddot{u}_i = \frac{Q_i}{\rho} \)

2. Stress and Plate-Displacement Relation
\[ M_{ij} + \partial_\alpha N_{ij} = Q_i - \rho \ddot{u}_i \]
\[ N_{ij} = G h \left[ (1 - \nu) \sigma_{ij} + \nu \left( \sigma_{kk} \delta_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \right) \right] \]

3. Plate Displacement Equation of Motion (Mindlin, 1951)
\[ \left( \frac{D}{2} \right) \left( \partial^2 \varphi_i + \lambda \partial_{xj} \varphi_i - \kappa \partial_{yj} \varphi_i \right) + \kappa \partial_{yj} \varphi_i = \frac{1}{\rho} \left( \partial^2 \varphi_i + \lambda \partial_{xj} \varphi_i - \kappa \partial_{yj} \varphi_i \right) \]

Cauchy’s Eq. of Motion(1828)
\[ \partial_\alpha \sigma_{\alpha i} + \rho \ddot{u}_i = \rho \ddot{u}_i \]

Generalized Hooke’s Law for Crystalline Solids
\[ c_{ij} \varepsilon_{ij} = \sigma_{ij} \]

Equation of Motion for Crystals
\[ \ddot{u}_i c_{ij} \partial_\alpha u_j + \rho \ddot{u}_i = \rho \ddot{u}_i \]

Higher Order Theories of Plates
\[ u_i (x, s, t) = \omega \varphi_i (x, s, t, t) \]
\[ u_i (x, s, t) = \pi (x, s, t, t) \]

Zeroth-Order (Extensional Motion)
\[ \dot{u}_i \dot{u}_i = u_i \dot{u}_i \]

First Order (Thickness Shear and Flexural Motion)
\[ \dot{u}_i \dot{u}_i = u_i \dot{u}_i \]
\[ \dot{u}_i \dot{u}_i = u_i \dot{u}_i \]
An Introduction to the Mathematical Theory of Vibrations of Elastic Plates
Epilogue

The Canterbury Tales, by Geoffrey Chaucer (1340-1400)

"...Of study took he utmost care and heed
Not one word spoke he more than was his need
And that was said in fullest reverence
And short and quick and full of high good sense
Pregnant of moral virtue was his speech
And gladly would he learn and gladly teach."

*Excerpt from “Raymond David Mindlin, A Biographical Sketch”, 1989 by H. Deresiewicz*